On “crystal growth” in harmonic space — James Tenney (1993-98)

It seems clear, intuitively, that a concern for harmonic coherence would lead to the use of relatively compact, connected sets of points in harmonic space, where “connected” simply means that every element is adjacent to at least one other element in the set. How might such compactness be defined more precisely? I have been investigating an interesting algorithm in which sets of points are chosen, one by one, in some n-dimensional harmonic space, under the condition that each new point must have the smallest possible *sum of harmonic distances* to all points already in the set. That is, at each successive stage in the growth of the lattice, the next ratio added to the set is one whose sum of harmonic distances to each ratio already in the set is minimal. There will be frequent “branch-points”, where two or more ratios have equally minimal HD sums, and here the choice might be random. Thus, for example, in a simple 2,3-space, and always beginning with a reference pitch (“1” in most of the figures below), the first new point chosen (pitch #2) can only be one of the four points marked “x” and “y” in Figure 1,

	x			
				$S(x) = \log_2(2) = 1$
y	1	y		
				$S(y) = \log_2(3) = 1.585$
	x			

Figure 1. (Base 2 logarithms are assumed throughout this paper).

and since S(x) is less than S(y), that second pitch must be at the octave above or below the reference pitch. If the upper octave is chosen (and it makes no difference to the final result which one is chosen, because the structure will remain invariant), the candidates for pitch #3 are the 6 adjacent points shown, with their corresponding sums of harmonic distances (S(x), S(y)), in Figure 2.

	x	
y	2	y
y	1	y
	x	

$S(x) = \log_2(2^3) = 3$

$S(y) = \log_2(2*3^2) = 4.17$

Figure 2.

Again at steps #4 and 5, the new pitches will be at the octave above or below those already in the set (because S(x) is still less than S(y) or S(z) as can be seen in Figure 3), but at step #6 this process of growth along the 2-axis will be replaced by an extension into the 3-dimension, as shown in Figure 4. Note that, in Figure 3, S(x) is still smaller than S(y), whereas, in Figure 4, S(x) is considerably larger than S(y), suggesting a gradually increasing tendency (as the number of pitches increases) toward extension into the 3-dimension, with a concomitant decrease in the tendency toward continued extension along the 2-axis, finally tipping the balance between the two dimensions at step #6. Figures 5 through 8 show the 2-dimensional lattices which result when this process is carried out through 10, 17, 24, and 36 points, respectively.

	x	
z	4	z
y	2	y
y	1	y
z	3	z
	x	

$S(x) = \log_2(2^{10}) = 10$

$S(y) = \log_2(2^4*3^4) = 10.34$

$S(z) = \log_2(2^6*3^4) = 12.34$

Figure 3. Step 5.

	x	
w	4	w
z	2	z
y	1	y
z	3	z
w	5	w
	x	

$S(x) = \log_2(2^{15}) = 15$

$S(y) = \log_2(2^6 * 3^5) = 13.92$

$S(z) = \log_2(2^7 * 3^5) = 14.92$

$S(w) = \log_2(2^{10} * 3^5) = 17.92$

Figure 4. Step 6.

4	10
2	7
1	6
3	8
5	9

Figure 5. 10 points in 2,3-space.

	17	
4	10	15
2	7	12
1	6	11
3	8	13
5	9	14
	16	

Figure 6. 17 points in 2,3-space.

	21	17	
23	4	10	15
19	2	7	12
18	1	6	11
20	3	8	13
24	5	9	14
	22	16	

Figure 7. 24 points in 2,3-space.

		(38)		
	21	17	34	
23	4	10	15	36
19	2	7	12	32
18	1	6	11	29
20	3	8	13	30
24	5	9	14	31
28	22	16	25	35
	27	26	33	
		(37)		

Figure 8. 36 (and 38) points in 2,3-space.

The symmetry of these sets is a characteristic property of all such “crystals” at certain stages of development, and is, in fact, one of the reasons why the analogy with crystal growth suggested itself for this process.

Consider now the specifically *musical* implications of these structures in 2,3-space: in Figure 9 the 17 points of Figure 6 are shown again with the numbers representing order of generation replaced by frequency ratios in Figure 9a and pitch names in Figure 9b (indexed for register, with C4 meaning middle C), with 1/1 shifted to the center of the lattice (the ratio 1/1 is identified here with F4, for no other reason than to *center* the whole pitch set with respect to the piano keyboard). In spite of the fairly large difference between $\log_2(2)$ and $\log_2(3)$, the lattice has only extended three octaves above and below the central point while two new pitch classes have been added to the set. This particular lattice is of special interest because it does not extend beyond the usual range of musical instruments, and could thus be mapped onto the piano keyboard (for example).

	8/1	
4/3	4/1	12/1
2/3	2/1	6/1
1/3	1/1	3/1
1/6	1/2	3/2
1/12	1/4	3/4
	1/8	

Figure 9a.

	F_7	
Bb_4	F_6	C_8
Bb_3	F_5	C_7
Bb_2	F_4	C_6
Bb_1	F_3	C_5
Bb_0	F_2	C_4
	F_1	

Figure 9b.

In the 24-element lattice shown in Figures 7 and 10a/b, a fourth pitch class has been added to the set, but the range has now been extended somewhat beyond that of the piano (a perfect fifth above and below the range of a Bösendorfer Imperial). It is at least very interesting (even if no more than a coincidence) that four of the five pitch classes of the Pythagorean pentatonic set are generated by this process before the pitch range has greatly exceeded the actual limits of musical perception.

	8/1	24/1	
4/3	4/1	12/1	36/1
2/3	2/1	6/1	18/1
1/3	1/1	3/1	9/1
1/6	1/2	3/2	9/2
1/12	1/4	3/4	9/4
	1/8	3/8	

Figure 10a.

	F_6	C_8	
Bb_3	F_5	C_7	G_8
Bb_2	F_4	C_6	G_7
Bb_1	F_3	C_5	G_6
Bb_0	F_2	C_4	G_5
Bb_{-1}	F_1	C_3	G_4
	F_0	C_2	

Figure 10b.

These lattices in 2,3-space have been considered here primarily to demonstrate some aspects of the “crystal growth” process in general, even though they may not be particularly significant, musically. The musical implications of this process become richer, and I believe clearly significant when we study the behavior of such gradually developing “crystals” in various higher-dimensional *pitch-class projection spaces*. For example, crystal growth in the 3,5-space seems closely related to the historical development of scales and tuning systems in Western music, from Pythagorean 3-limit through 5-limit just systems, and even including our more recent 12-tone equal temperament. Beginning, as before, with an initial reference pitch, the lattice grows along the 3-dimension, linearly, until four of the five elements of a Pythagorean pentatonic set have been generated, as shown in Figure 11 (note: from here on, the 3-dimension is again represented by horizontal axes, but the vertical axes now represent the 5-dimension).

	z	x	x	z		
						$S(x) = \log_2(3^4 * 5^4) = 15.63$
y	3	1	2	4	y	
						$S(y) = \log_2(3^{10}) = 15.85$
	z	x	x	z		
						$S(z) = \log_2(3^6 * 5^4) = 18.80$

Figure 11.

At this stage, there are very nearly equal values of harmonic distance sums for two different sets of candidate pitches, the pair labeled "y" in Figure 11, either of which would result in a full pentatonic set, and the four points labeled "x", any one of which will initiate an extension into the 5-dimension (and it is, of course, one of these that <u>is</u> chosen by the algorithm). If a "y" had been chosen instead of an "x", the harmonic distance sums at the <u>next</u> stage would have been as shown in Figure 12

	w	z	x	z	w	
y	5	3	1	2	4	y
	w	z	x	z	w	

Figure 12.

where

$S(x) = \log_2(3^6 * 5^5) = 21.12$
$S(y) = \log_2(3^{15}) = 23.77$
$S(z) = \log_2(3^7 * 5^5) = 22.70$
$S(w) = \log_2(3^{10} * 5^5) = 27.46$

and it is especially to be noted that even S(z), as well as S(x), is smaller the S(y) (which latter, if it had been chosen, would have extended the Pythagorean set beyond the pentatonic set).

Thus, in addition to other reasons that might be adduced for the ubiquitous manifestations of the Pythagorean pentatonic scale (not only in Western music, but in many other cultures as well), this crystal growth model suggests another – specifically harmonic – reason: for a set of pitches arrayed along this single (3-) axis, five pitches constitute a kind of limit, beyond which the tendency toward extension into a new dimension (the 5-dimension) becomes decisive. This assumes, of course, that such an extension is not prohibited by Aristotelian dogma, as it evidently was in Western music theory until sometime in the 16th century.

The Pythagorean pentatonic set may be conceived as a pitch set which arises when extension into the 3,5-plane is just slightly *delayed* beyond the point where the algorithm would have begun that extension. Interesting sets also arise when the extension into the 3,5-plane occurs “prematurely”, as shown in Figures 13a and 13b, below. Note that these represent the two Japanese Koto scales, “Hirajoshi” and “Kumoijoshi” (Malm, date, p. 178), and the latter might even be taken as approximated by the Balinese 5-tone *pelog* scale (McPhee, date, p. 47).

1/1	3/2	9/8
8/5	6/5	

Figure 13a.

	1/1	3/2
16/15	8/5	6/5

Figure 13b.

Once the lattice (as generated by the algorithm) has begun to move into the 3,5-plane, the following symmetrical configurations are generated, containing 8, 12, and 14 pcs, respectively (Figures 14-16).

5/3	5/4	15/8	45/32
4/3	1/1	3/2	9/8

Figure 14. 8 points in 3,5-space
(note that this contains both major and minor diatonic sets; major on 1/1 (if 45/32 is omitted), and minor on 5/4 (if 4/3 is omitted).

5/3	5/4	15/8	45/32
4/3	1/1	3/2	9/8
16/15	8/5	6/5	9/5

Figure 15. 12 points in 3,5-space (this contains both the major and the minor sets simultaneously, both built on 1/1 (the two upper rows minus 45/32 for the major; the two lower rows minus 16/15 for the minor).

	5/3	5/4	15/8	45/32	
16/9	4/3	1/1	3/2	9/8	27/16
	16/15	8/5	6/5	9/5	

Figure 16. 14 points in 3,5-space (adding to the 12-set of Figure 15 two of the most frequently needed “alternative tunings”, for the major sixth – 27/16 (as the fifth of a secondary dominant) in addition to 5/3 (the submediant) – and the minor seventh – 16/9 (as the subdominant of the subdominant) in addition to 9/5 (the third of a dominant minor)).

At this stage in the “crystal growth” process, if ratio-generation is not constrained to remain within the 5 limit, the next element chosen by the algorithm will be one of the 7-ratios indicated by the points labelled “x” in Figure 17, so the 14-element 3,5-lattice appears to be approaching some kind of natural limit for 5-limit lattice structures, just as the four-element Pythagorean set seemed to be doing for 3-limit structures.

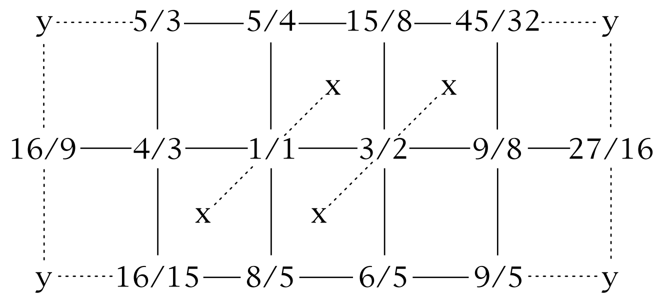

Figure 17. Candidates for the next element to be added to the lattice of Figure 16, where S(x) = 84.82, S(y) = 87.98.

Figures 18 and 19 show symmetrical lattices of 18 and 22 points, respectively, in 3,5,7-space.

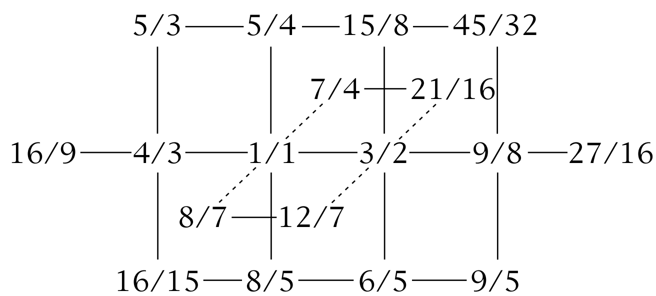

Figure 18. 18 points in 3,5,7-space.

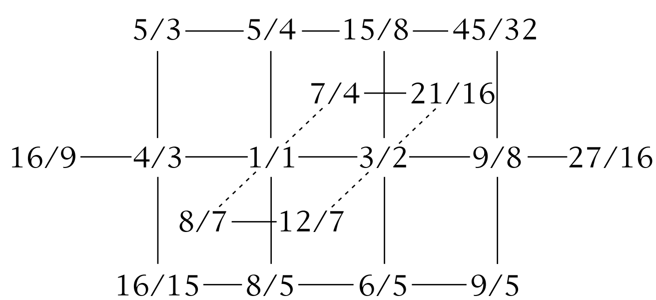

Figure 19. 22 points in 3,5,7-space.

Continuing in this way, larger and larger lattices will be built up, but for some reason it appears that none of them are completely symmetrical again until 76 points have been generated, although a few of them are very *nearly* symmetrical. After 50 elements have been generated in 3,5,7-space – *if* ratio-generation is not constrained to remain within the 7-limit – the next element chosen by the algorithm will be one involving the next-higher prime number – 11 – thus initiating growth in a new dimension.

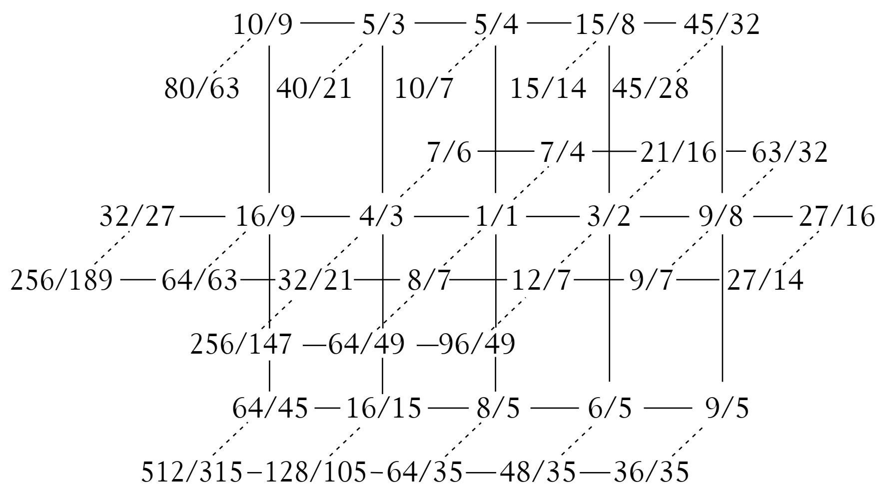

Figure 20. 41 points in 3,5,7-space.