A NOTATION SYSTEM FOR EXTENDED JUST INTONATION

2003

Extended just intonation is a kind of ratio-scale tuning based upon pure intervals—i.e., those analogous to the pitch relationships of the harmonic series. To handle the complex ratio relations necessary to achieve a truly accurate just tuning of modulatory triadic music, plus its many harmonic extensions beyond simple triads, a basic notation as close as possible to familiar, widely used Western music notation is desirable. As long as certain fundamental differences in what the symbols refer to are clearly understood, there is no ambiguity in such a procedure.

This notation is not tied to any particular diapason (such as A = 440 Hz, A = 435 Hz, or even a C-based or G-based tuning system). What remains constant are the ratio relations between pitches. The pitch reference of notes without any accidentals is not the tempered scale commonly presumed to be the basis of Western music, but rather a tuning of the seven-tone major scale resulting from tuning the three primary major triads (tonic, dominant, and subdominant) in the ratio 4:5:6. For this reason the note names are subordinated to C, since the key signature of C major has no sharps or flats. The ratio relationship to C of the scale notes is 1/1 (C), 9/8 (D), 5/4 (E), 4/3 (F), 3/2 (G), 5/3 (A), 15/8 (B), 2/1 (C). The sequence of adjacent-pitch ratios is 9/8, 10/9, 16/15, 9/8, 10/9, 9/8, 16/15 (Figure 29).

This results in a scale in which there are two sizes of whole step (9/8, 10/9) and one size of half step (16/15). The interval of difference between the two kinds of whole step is 81/80, the comma of Didymus. This interval is represented in notation by the accidentals + (raise) and - (lower). To turn the interval C, D into a 10/9, it must be notated C, D- or C+, D. To turn the interval D, E into a 9/8, it must be notated D, E+ or D-, E (Figure 30).

In examining this just major scale for its further triadic possibilities, let us keep in mind that a just minor triad is the exact inversion of a just major triad. To add intervals (e.g., C, E plus E, G equals C, G) you must multiply ratios (correspondingly, 5/4 multiplied by 6/5 equals 3/2). Dealing with a

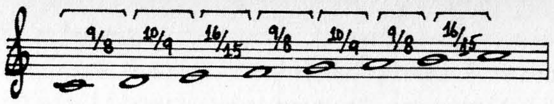

FIGURE 29. Uninflected notational symbols, referring to triadic just intonation (2, 3, 5 tuning) of the diatonic major scale of C: in ratio terms the pitches are 1/1–9/8–5/4–4/3–3/2–5/3–15/8–2/1.

minor triad, begin with its top note (its fifth) and work downward, using the same pair of ratios: 5/4 for the (upper) major third and 6/5 for the (lower) minor third. If you attempt to do this with the D-minor triad (supertonic triad), you discover that although the major third (F, A) is in the ratio 5/4, the minor third (D, F) is not in the ratio 6/5, but 32/27. The pure tuning of the supertonic triad is thus D-, F, A (10/9–4/3–5/3). A further example: the intervals B, D and D, F are in ratio terms 6/5 and 32/27. For the resulting triad to become a just minor triad, not only would F have to become F♯, but an additional + would be necessary: F♯+ (in ratios, 15/8–9/8–45/32). It is in this way that the syntonic comma (comma of Didymus) is used to keep both kinds of triads in just consonant (simple ratio) tuning.

When triadic polychords are formed by piling up triads, as was increasingly the case in nineteenth-century European concert music, leading eventually to early twentieth-century polytonality, the intention to create a just tuned version of such a style leads to complex tuning and often involves deliberate use of just such dissonant versions of thirds as those replaced by consonant versions (as described above). Jazz harmonies of the big-band era, from the 1920s through the 1940s, are an especially rich area for such

FIGURE 30. Examples of the use of the syntonic comma symbols + and -, symbolizing respectively the raising and lowering of a pitch by 81/80. Taking C as 1/1, the four intervals are 1/1–10/9; 81/80–9/8; 9/8–81/64; 10/9–5/4.

tuning, even without the inclusion of the "natural" seventh, so evocative of blues tuning. Take, for example this chord: F, C, E♭, G, B♭, D, F. If tuned so that all the interlocked triads are just, the high F must be F+, which means that between the low F and the high one there is an interval of a double octave, sharp by one syntonic comma (Figure 31).

The way the simple consonant ratios in just tuned triadic music relate to each other can most easily be seen by studying the layout of pitches in the 2, 3, 5 lattice presented in my article "Rational Structure in Music."[1] (See Figure 23.)

Many string quartets use an electronic tuner to achieve accuracy in microtonal inflections. Since such tuners are designed to show deviations from a twelve-tone equal-tempered scale, this can easily result in distortions when applied to extended just intonation unless this process is altered so as to allow that the notes without accidentals in the notated C-major scale *do not* refer to equal-tempered tuning but rather to the tuning described above. The tuning of the open strings which extended just intonation uses is exactly the same as that tuning used by virtually all string quartets: four notes each a just perfect fifth (3/2, or 702 cents) apart. Since A is the traditional tuning note, A and E require no accidentals. But in the just C-major scale used in extended just intonation, the fifth between D and A is one syntonic comma too small, and so the D needs to be tuned as D-. This means also that the interval G, D must be notated as G-, D- (Figure 32). If the comma inflections are applied to tempered pitches, a distorted tuning results. It is important to remember that this is identical to normal tuning: it merely looks unfamiliar. In a just tuned C-major triad involving open G strings (on the violins) and

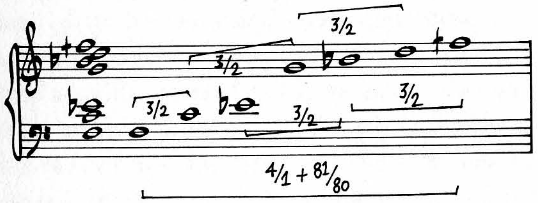

FIGURE 31. An example of the syntonic comma in practice: in this chord all the interlocked triads are just, resulting in a syntonic comma discrepancy between the low F and the high F.

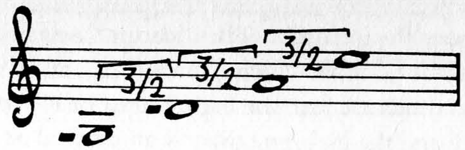

FIGURE 32. The notation of the open strings of a violin in Johnston's system. Because an uninflected A on the staff represents the ratio 5/3 (to C 1/1) and an uninflected D the ratio 9/8, the D needs the syntonic comma symbol (the minus) to bring it down to 10/9, a pure fifth below the 5/3; likewise, the G needs the same symbol to make it the ratio 40/27, a pure fifth below the 10/9.

open C and G strings (on viola and cello), the open E strings of the violins will not be in tune with the just major third of the chord: they are one syntonic comma too high. This shows in the notation: the names of the notes in the C-major triad will be C-, E-, and G- because the open C string and G string are being used as root and fifth of the triad.

What characterizes mainstream Western concert music since the time of *Ars Nova* (the fourteenth century) more than any other aspects is, first, its rhythmic freedom from repetitive patterns; second, its organization into contrapuntal phrases punctuated by recognizable cadences; and, third, its gradual but definite tendency toward triadic consonant harmonies, growing out of the common practice for handling consonant and dissonant intervals between all pairs of contrapuntal parts. As long as vocal music predominated over instrumental, the question of compromise tunings (various kinds of temperaments) did not loom large. The pitch notation of Renaissance music allowed for ♯ and ♭ inflections, gradually emerging from the performance practice of *musica ficta,* but no symbol was provided for the syntonic comma, doubtless because its microtonal size (less than a tempered eighth tone) made it unusable as a melodic interval, and its extreme dissonance (reflected in its complex ratio 81/80) made it unusable even as a nonharmonic dissonance. Its importance as an adjustment to keep triadic harmonic combinations purely in tune emerged only as harmonic progressions began to play a conscious and deliberate part in the design of compositions. This coincided with the increasing use of instrumental accompaniment or even instrumental compositions independent of vocal music.

The use of consonant major and minor thirds (5/4 and 6/5, respectively)

began to predominate in Baroque music even over the traditionally inviolable consonances of the perfect fifth and the perfect fourth (3/2 and 4/3), leading to various forms of meantone temperament. Insisting upon both kinds of consonances would have made the conscious, deliberate use of adjustments by syntonic comma unavoidable. It was—above any other consideration—the increasing use of keyboard instruments, not merely for rehearsal but in performance ensembles, that made the search for ever more complex kinds of temperament inevitable as a wider and wider range of modulatory harmonic progression became a goal. During this transitional period and the gradually emerging Classical era, the tuning of keyboard instruments was still in the hands of the performer. With the introduction of the pianoforte, the tuning of the instrument passed to specialist tuners, removing the task of keeping music in tune from the pianist's responsibilities. It is this transition into Romantic music that brought the complex tuning of twelve-tone equal temperament finally into use.

If one wants to reconceive the music of these stylistic eras in terms of extended just intonation, it is first necessary to make limited and very specifically prescribed use of nonretunable instruments, and second to consider what adjustments are really possible for other instruments and whether performers of those instruments will be able, willing, and motivated to make these changes in performance practice. The easiest adjustments are those of bowed string instruments, but the orchestral practice of having large numbers of string players performing in unison works powerfully against this, as does the ubiquitous use of vibrato by string players.[2]

While it is possible theoretically to conceive of piling up powers of prime numbers greater than 5, as a truly functional additional lattice dimension makes possible, the practical usefulness of this is actually quite limited.[3] Piled-up perfect fifths (or more commonly perfect fourths) and progressions based upon chord root movement up or down by perfect fifth or fourth are, of course, commonly used in Western music. Piled-up major thirds are commonly used only in pairs (the augmented triad), but root movement by major thirds or by minor thirds is much more common, leading rather quickly in both cases to what in extended just intonation can only be called enharmonic nonequivalents. C, E, G♯, B♯ is the simplest of such progressions. C, E♭, G♭, B♭♭, D♭♭ is almost as simple. Both these progressions, in their much tamer twelve-tone equal-tempered versions, are standard fare in nineteenth-century Romantic music, notably in much music by Liszt or Wagner. An earlier use of such progressions is in the secular choral music of the High

Renaissance, notably in music by Gesualdo. It is hard to imagine using the prime number 7 in an analogous way, let alone 11 or 13 or even higher primes. (Not that it has not been done: cf. Toby Twining's *Chrysalid Requiem.*) By far the commonest earlier use suggesting such intervals is in complex chords approximating overtone combinations in the music of Debussy and other Impressionist composers. Again the effect of temperament is not only to distort the tuning of such "overtones" but to tame down the expressive impact they have. Call to mind the discussion above of the use of such harmonies in big-band jazz. The performance practice of jazz presumes the inflection by ear of such chords, at least in ensembles capable of such flexibility.

For notation purposes, each higher prime number entails the introduction of a pair of symbols indicating the interval difference between the overtone indicated by the new prime number and the just tuned note in the 2, 3, 5 system nearest in pitch to it which has a simple notation. (This specification will be explained below.) When practical, the number itself is used for the sign belonging to the overtones, for instance 7. The retrograde inversion of this symbol, ∠ in this case, is used for the sign belonging to the inverted series (analogous to "undertones"). In C major this chord is C, E, G, B7♭, because B♭ (9/5) is the nearest simply notated pitch in the 2, 3, 5 system. B♭- (16/9) is nearer in pitch, but its accidental involves an extra symbol (-). When the symbol 7 is used with a ♭, the stem of the ♭ is used as the vertical of the 7 and a composite of the two symbols results: 7♭. In E major the notation is simpler: E, G♯, B, D7. In E minor (remember, the inversion puts B at its top) the notes are B, G, E, C♯∠. When the symbol ∠ is used with a ♯, the right vertical of the ♯ is used as the vertical of the ∠, and a composite symbol again results: ♯∠. In C minor (remember, the inversion puts G at its top) the notation is simpler: G, E♭, C, A∠. If the symbol 7 is combined with a ♯, the left vertical of the ♯ is used as the vertical of the 7, and a composite symbol results: 7♯. If the symbol ∠ is combined with a ♭, the vertical of the ♭ is used as the vertical of the ∠, and a composite symbol results: ♭∠.

In moving around in the harmonic space generated by a multidimensional tonal lattice, many relationships not easily understood from a single overtone series can be observed to have an easy intelligibility to the ear even though they may seem complex when described intellectually. An easy way to understand this is to consider just tuning the C-major and C-minor triads. Then add the seventh partial (B7♭) to the major and the sub-seventh partial (A∠) to the minor. If one makes a progression from one of these chords to the other, the C and G are common tones. The thirds, E and E♭, are a chromatic half

step (25/24) apart. (5/4 divided by 6/5 is 25/24.) The interval between the two sevenths (B7♭ and A∠ refer respectively to 7/4 and 12/7) is 49/48 (7/4 divided by 12/7 is 49/48). Since, in the overtone series, 25/24 in the next octave of partials is 50/48, we can observe that the chromatic half step is split, in the overtone series, by the 49/48 interval. In the overtone series of C, 25 is the major third of the major third, and 49 is the natural seventh of the natural seventh. (Imagine the chord C, E, G, B7♭, D7, F7+, A77♭+, or the same chord in the overtone series of A: A, C♯, E, G7, B7, D7, F77+. This is a complex chord and would be difficult to hear with accuracy. But in the case of moving from C major with its natural seventh to C minor with its natural sub-seventh, both of these relationships are present and are not difficult to tune by ear.) This is an example of chromaticism in extended just intonation. We are dealing here with a progression which is an example of what Lou Harrison called "free style" composing: composing not restricted to a single overtone series nor to a predetermined choice of notes to be used (as with the use of a mode or a raga).

In the case of the number 11, using the Arabic numeral is not practical. The eleventh partial of the overtone series is often used by piano tuners needing to tune two pianos a quartertone apart, because it is quite close to halfway between the perfect fourth and the tritone. This points up a characteristic of intervals involving 11: they have an ambivalence (e.g., 11/9 sounds like a neutral third, neither major nor minor). This ambivalence suggested using the arrow, which has sometimes been used in twentieth-century music to indicate inflection by quartertone. Up-arrow raises; down-arrow lowers. The arrow is used to alter the perfect fourth, rather than one of the several just tritones, the notation for any one of which involves at least two symbols. In the overtone series of C, the 11th partial is F↑. In the undertone series beginning on C (F minor) the sub-eleventh (16/11) is G↓. The neutral third, prominent in some folk musics, can be represented in just tuning by the 11/9 ratio. In a neutral A triad derived from A major, the neutral third would be notated as C↑- (since A is the 9/8 above G-), the ratio 55/54. Alternatively, this neutral triad could be derived from A minor, in which case (since E is a 9/8 below F♯+) the neutral third would be notated as C♯↓+. This latter is a much more difficult tuning to achieve by ear.

As an example, consider the opening statement of the folk hymn "Lonesome Valley" in my *String Quartet no. 5*. The hymn tune, traditionally in a pentatonic scale, is here tuned so that the minor thirds in the pentatonic scale are neutral thirds (Figure 33). It was the major mode derivation that was used in this passage. Here the aim was not to create a neutral triadic harmony but

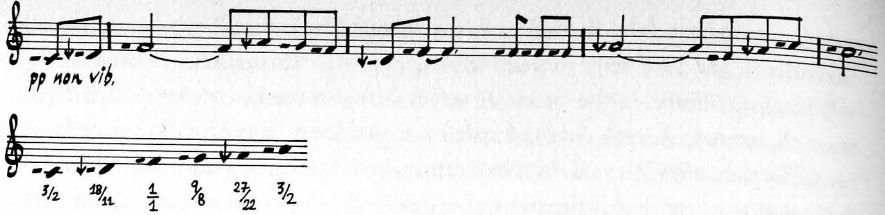

FIGURE 33. The opening statement of the folk tune "Lonesome Valley" in *String Quartet no. 5,* and the pentatonic scale on which it is based. With F- considered to be 1/1, the third and fifth degrees of the scale are eleventh-partial relationships, respectively 27/22 and 18/11. Copyright by Smith Publications, 2617 Gwynndale Ave., Baltimore, Maryland, USA. Used by permission.

to present thirds that defy identification as major or minor. In the course of this quartet, a number of different tunings of this hymn tune are presented, in a meditation on its many possible emotional meanings (Figures 34 and 35).

The interval made between the third octave of the fundamental of an overtone series and its thirteenth partial, the ratio 13/8, is approximately 27 cents larger than the just minor sixth. Because the resulting interval is much closer in size to a minor sixth than to a major sixth and the minor sixth is seldom used as an "added sixth" in common Western musical practice, the 13/8 interval is much more likely to occur as an "under-third" (a 16/13) placed below the root of the chord. In C major this chord will be A13♭, C, E, G. In this case the numeral itself is used as the accidental. In C minor the equivalent

FIGURE 34. Transformation of the "Lonesome Valley" melody in *String Quartet no. 5,* employing a 7–limit scale. Copyright by Smith Publications, 2617 Gwynndale Ave., Baltimore, Maryland, USA. Used by permission.

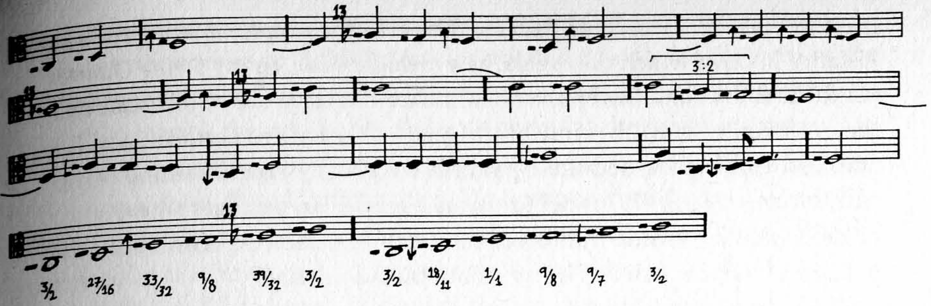

FIGURE 35. Further transformations of "Lonesome Valley" from *String Quartet no. 5;* the scales used successively in this viola passage are a 13–limit and an 11–limit one. Copyright by Smith Publications, 2617 Gwynndale Ave., Baltimore, Maryland, USA. Used by permission.

chord would be C, E♭, G, B^{ƐI}. (The last symbol here is meant to represent a 13 rotated laterally 180 degrees. This added "major seventh" is verbalized "B sub-thirteen.") The scale made by the eighth through the sixteenth partials of an overtone series (in C: C, D, E, F↑, G, A13♭, B⅂♭, B, C) can be used as an eight-tone mode, or as a tone cluster, or as a widely voiced chord in the manner of big-band jazz harmonies. Any of these manners of use are not difficult to learn how to tune by ear. It is hard to imagine how the thirteenth of the thirteenth could be made musically meaningful, so very limited use of a complete 13–limit matrix would be practicable.

The sizes of the successive ratios in the overtone octave between the eighth and the sixteenth partials gradually decrease from a whole tone (9/8) to a half tone (16/15). In the next overtone octave, between the sixteenth and thirty-second partials, the sizes of the successive intervals gradually decrease from a different form of half tone (17/16) which, at 105 cents, is very close in size to a tempered half step, to a quartertone (32/31) which is very close in size to a tempered quarter step. The ratio 19/16 (298 cents) is easily regarded as like an equal-tempered minor third. The prime number ratio 23/16 is slightly sharper than an equal-tempered tritone. The prime number ratio 29/16 is slightly sharper (by 6 cents) than an equal-tempered minor seventh.

The new prime-numbered partials in the overtone series between the sixteenth and the thirty-second are not easy to tune by ear harmonically, especially in a musical context of just tuned intervals clearly based upon intervals formed from lower prime number relationships. But a melodic use of these intervals is a learnable pattern, the performer approximating the size distance in cents which they make as a melodic scale. In my *String Quartet no. 9*, the fourth and final movement makes extensive use of this scale and the inversion of it. At the time of the quartet's premiere performance, the Stanford Quartet learned these patterns entirely by experimenting by ear without any electronic model to use as a guide. (In passages where some of the unfamiliar prime-numbered intervals were used experimentally, the performers were much less secure about their accuracy.)

The seventeenth partial is regarded in the notation as an altered chromatic half tone (25/24). Thus the symbol 17 raises the notated pitch by the ratio 51/50 (34 cents or about a sixth tone). The ratio 17/16 is thus notated as C♯17. Since the 17th partial is commonly used as a minor ninth added to a just tuned dominant seventh chord, this notation is an unfamiliar one and may seem arbitrary in that usage, but the complication of symbols involved in using D♭- (the diatonic half-tone) argued against using it, and the distance in cents from D♭ plus the fact that D♭ is used far less often in just tuned equivalents of familiar progressions argued against using it. The inversion of this symbol is.ㄥI

The nineteenth partial is regarded in the notation as an altered minor third. It lowers the pitch by 17 cents. The ratio 19/16 is thus notated as E♭6I. Its inversion, which raises the pitch by the same amount, is symbolized by 19. To avoid the impression that this first symbol is the number sixty-one, the 9 of the 19 should be written with a diagonal straight line (as in many people's script writing of this numeral), so that when its retrograde inversion is used it does not look like the numeral six.

To notate the ratio 23/16, it is the augmented fourth 45/32 (F♯+, a major third above the supertonic 9/8) which is altered by the symbol 23. The left side of the bottom of the 3 (in the 23 symbol) is attached to the upper end of the left vertical of the ♯. To notate the inversion of this interval, the ratio 32/23, the note altered is the diminished fifth 64/45 (a major third below the minor seventh 16/9). In the key of F minor (figuring down from C) this is the G♭- which is a major third below B♭-. The retrograde inversion of the symbol 23 is easily drawn by hand. It looks like an E rounded to resemble a cursive script capital E followed by a 2 rotated up 180 degrees so that the horizontal straight line is on the top. The end of the curve of the rotated 2 is attached to the vertical of the ♭ symbol: ƐZ♭.

To notate the ratio 29/16, it is the minor seventh 9/5 (a minor third above a perfect fifth) which is altered by the symbol 29. As with the symbol 19, the

9 is not drawn to look like an inverted 6, but with a diagonal straight line. In the overtone series of C this interval is notated B♭ with the diagonal of the 9 (of the symbol 29) attached to the vertical of the ♭. To notate the inversion of this interval, the ratio 32/29, the note altered is the major second 10/9 (a syntonic comma smaller than the major second 9/8). In the key of F minor (figuring down from C), this note is D-, a minor third below F. The retrograde inversion of the symbol, 6Z, is easily drawn by hand. It looks like the inverted 9 (6) drawn as described above, followed by the rotated numeral 2 (already described above). The end of the curve of the inverted 2 is attached as before (see above).

To notate the ratio 31/16, it is the major seventh 15/8 which is altered (a major third above a perfect fifth) which is altered by the symbol 31. In the overtone series of C this interval is notated B31. To notate the inversion of this interval (32/31), the ratio altered is 16/15 (a major third below the perfect fourth). In the key of F minor (figuring down from C), this note is D♭-, a minor third below F. The retrograde inversion of this symbol is IƐ with the Ɛ notated like the cursive capital E (rounded like the numeral 3).

The notation symbols may be combined into composite symbols to save space in notating accidentals, except that the syntonic comma (+ and -) is never combined with any other symbol and always comes next to the note itself.

The systematic shown in Figure 36 can be expanded or contracted to include

raise	lower	ratio	cents	amount by which...	exceeds
♯	♭	25/24	71	5/4	6/5
+	−	81/80	22	9/8	10/9
ㄥ	7	36/35	49	9/5	7/4
↑	↓	33/32	53	11/8	4/3
13	ƐI	65/64	27	13/8	8/5
17	ㄥI	51/50	34	17/16	25/24
6I	19	96/95	18	6/5	19/16
23	ƐZ	46/45	38	23/16	45/32
29	6Z	145/144	12	29/16	9/5
31	IƐ	31/30	57	31/16	15/8

FIGURE 36. Notational symbols devised by Johnston for just intervals through the 31–limit, showing their ratio value, their size in cents, and their derivation.

any combination of prime numbers, and it is not necessary to have reference to tonal lattices or even to ratios, once the intrinsic meaning of the symbols used and their degree of modification of the pitches they precede is understood. If it is understood not only in theory but by ear, the free invention of a modulatory music in just intonation extended to any degree becomes a practical undertaking. It is not the mathematics that makes music sound in tune: it is the performers following leads from composers who themselves understand how to design the music so that this clarity of vibration relations results.

NOTES

1. Cf. "Rational Structure in Music." The notes near to each other on the lattice are more easily tunable by ear than those which lie farther apart. Remember, in studying the lattice, that notes diagonally adjacent are not harmonic neighbors but can be reached only by using straight verticals or horizontals and right-angle changes of direction. The way to make dissonant intervals intelligible to the ear is to connect them by a path of easily tuned consonances on the lattice as the F and F+ were connected in the example given above.

2. Even very commonly used progressions in music, such as I, vi, ii, V, I, have hidden problems of intonation. Look at this progression in C major. Chord I is C, E, G; with common tones C and E it moves to A, C, E; with common tone A this moves to D, F, A, but the interval D, A in the extended just C-major scale is not a true perfect fifth, needing to become D-, A; with common tone D- this moves to G-, B-, D-, and with common tone G-, moves on to C-, E-, G-. The entire progression has not moved in a circle, returning to C, but in a spiral, flatting in pitch by a syntonic comma (about 21.5 cents). This amounts to movement on the 2, 3, 5 tuning lattice to a new location (and a new notation) for C. (Cf. my description of lattices in "Rational Structure in Music.") Since there is a contradictory tradition (in the majority of Western concert music) of returning at the end of a composition to the same tonic note as at its beginning, the progressions of a composition have to be very carefully designed to make a "round trip" on the lattice, and the movement the musical progressions make on the lattice needs very conscious planning.

3. Each axis on the lattices presented in "Rational Structure in Music" represents the ratio relation of each prime number and successive powers of it to the number 2 and powers of it. Rather than 0 at the center and 1, 2, 3, 4, 5, etc., mirrored by -1, -2, -3, -4, -5, etc., as is the case in a familiar graph, these integers are instead exponents. So we have, on the 3–axis, in the center, 3 to the 0 power and 3 to the successive powers 1, 2, 3, 4, 5, etc., mirrored by 3 to the successive powers -1, -2, -3, -4, -5, etc. This process applies similarly to each successive prime number. It should be immediately apparent that this leads to a multidimensional system impossible to represent adequately on the two-dimensional surface of a page.